A Virtual Working Format for Thermomechanics

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Motivation

Hodiernal continuum mechanics is *multiscale* and *multiphysics*:

- ***interdependent phenomena take place at different scales* in bodies regarded as an *interactive composition of material structures*;**
- **plurality of scales and material structures calls for *adjustments in the standard modeling format*: primarily, to lay down in a systematic manner the relevant balance and inbalance laws, making sure that no interaction is overlooked.**

Plan

To present a format modeled after the *virtual working format*, using *thermomechanics as a paradigmatic example*.

The Standard Mechanical & Thermal Structures

- ***mechanical body structure*** **based on**
  - **Virtual Working Principle**

$$\int_{\mathcal{P}} \mathbf{S} \cdot \nabla \mathbf{v} = \int_{\mathcal{P}} \mathbf{d} \cdot \mathbf{v} + \int_{\partial \mathcal{P}} \mathbf{c} \cdot \mathbf{v}\,, \quad \mathbf{v} \in \mathcal{V},\ \mathcal{P} \subset \mathcal{B}\,,$$

  implying force balance ;

- **thermal body structure based on**
  - **energy balance**

$$\dot{\varepsilon} = -\mathbf{div}\,\mathbf{q} + r\,,$$

  - **entropy imbalance**

$$\dot{\eta} \geq -\mathbf{div}\,\mathbf{h} + s, \quad \textbf{with} \;\; \mathbf{h} = \vartheta^{-1}\mathbf{q}, \;\; s = \vartheta^{-1} r\,.$$

Body as a Composition of Mechanical & Thermal Structures. i. Kinetics

- **kinetic variables are**
  - **mechanical displacement** $\mathbf{u}$**, with**

$$\mathbf{v} = \dot{\mathbf{u}} \equiv \textbf{velocity},$$

  - **thermal displacement** α**, with**

$$\vartheta = \dot{\alpha} \equiv \textbf{temperature},$$

  defined over ***space-time cylinder*** $\mathcal{B} \times (0, T)$

- **process** $(x, t) \mapsto \big(\mathbf{u}(x, t), \alpha(x, t)\big)$**, with**

$$\big(\mathbf{v}(x, t), \vartheta(x, t)\big) \equiv \textbf{realizable velocity pair}$$

- $\big(\delta\mathbf{u}(x, t), \delta\alpha(x, t)\big) \equiv$ **virtual velocity pair**

Thermal Displacement, a Few References

- **M. von Laue**, *Relativitätstheorie*, Vol. 1 Vieweg, Braunschweig (1921). (for von Laue, **thermacy** = *minus* **thermal displacement**)
- **A.E. Green** and **P.M. Naghdi**, Thermoelasticity without energy dissipation. *J. Elasticity* 31 (1993), 189-208.
- **C. Dascalu** and **G.A. Maugin**, The thermoelastic material-momentum equation. *J. Elasticity* 39 (1995), 201-212.
- **P. P-G** and **A. Tiero**, Un formato tipo lavori virtuali per la termodinamica dei processi omogenei. Proc. XIV Congr. Naz. Meccanica Teor. Appl. (Como, Italy - October 1999).

Body as a Composition of Mechanical & Thermal Structures. ii. Dynamics

For $\mathcal{P}$ a *subbody* of $\mathcal{B}$, and $I = (t_i, t_f)$ a *subinterval* of $(0, T)$, dynamics specified by

(i) *internal virtual working*

$$\delta\mathcal{W}^{(i)} = \int_{\mathcal{P}\times I} (\mathbf{s}\cdot\delta\mathbf{u} + \mathbf{S}\cdot\nabla\delta\mathbf{u} + h\,\delta\alpha + \mathbf{h}\cdot\nabla\delta\alpha)\,,$$

where

- s **and** S $\equiv$ $0-$**th and** $1-$**st order mechanical interactions**
- h **and** h $\equiv$ $0-$**th and** $1-$**st order thermal interactions**

Note

Treatments of mechanical and thermal entities should be kept *as parallel as possible*. But, *parallelism broken by*

- ***invariance requirements***

  in a galilean observer change,

$$\mathbf{v} \mapsto \mathbf{v}^{+} = \mathbf{v} + \mathbf{t}, \quad \alpha \mapsto \alpha^{+} = \alpha.$$

  Thus, translational invariance of $\delta\mathcal{W}^{(i)}$ implies a symmetry-breaking conclusion:

  the 0th order stress s ***is null*.**

- ***habit***

habitual entropy flux* = *minus the 1-st order thermal interaction h**.**

(ii) ***external virtual working***

$$\delta\mathcal{W}^{(e)} = \int_{\mathcal{P}\times I} (\mathbf{d}\cdot\delta\mathbf{u} + \mathbf{p}\cdot\delta\dot{\mathbf{u}} + s\,\delta\alpha + \eta\,\delta\dot{\alpha}) + \int_{\partial\mathcal{P}\times I} (\mathbf{c}\cdot\delta\mathbf{u} + c\,\delta\alpha) + \int_{\mathcal{P}\times\partial I} [\![\mathbf{p}\cdot\delta\mathbf{u} + \eta\,\delta\alpha]\!]$$

- $(\mathbf{d},\mathbf{c})$ **and** $(s,c) \equiv$ ***distance*** **and** ***contact*** ***interactions*****,** ***mechanical*** **and** ***thermal*****;**
- $\mathbf{p} \equiv$ **momentum,** $\mathbf{Sn} \equiv$ **momentum flux;**

  $\mathbf{d} \equiv$ **momentum source** $\equiv$ **(*noninertial* distance force);**
- $\eta \equiv$ **entropy,** $\mathbf{h}\cdot\mathbf{n} \equiv$ **entropy flux,** $s \equiv$ **entropy source;**
- $\mathbf{c} \equiv$ **contact force,** $c \equiv$ **contact heating;**
- $\mathbf{p}_f, \ldots, \eta_i \equiv$ ***external*** ***actions*****, mechanical & thermal,** ***at time boundaries*** **of** $\mathcal{P}\times I$ **:**

$$\int_{\partial I} [\![\mathbf{p}\cdot\delta\mathbf{u} + \eta\,\delta\alpha]\!] := \mathbf{p}_f(x)\cdot\delta\mathbf{u}(x,t_f) + \eta_f(x)\,\delta\alpha(x,t_f) + \mathbf{p}_i(x)\cdot\delta\mathbf{u}(x,t_i) + \eta_i(x)\,\delta\alpha(x,t_i)\,.$$

The Virtual Working Axiom

(VW) ***The <u>internal and the external working should be equal</u>:***

$$\delta\mathcal{W}^{(i)} = \delta\mathcal{W}^{(e)},$$

for each virtual velocity pair defined over the closure of any subcylinder $\mathcal{P} \times I$ ***of*** $\mathcal{B} \times (0, T)$ ***and such as to vanish at the end of*** I ***itself.***

Notes

- **Just as** α **called** ***thermal displacement*** **to allude to role analogy with mechanical displacement** $\mathbf{u}$**, why not to call** η ***thermal momentum*****, by analogy with the mechanical momentum** $\mathbf{p}$**?**

- **Just as** $\mathbf{p}$ **thought of as measuring** ***reluctance to quiet*****, why not to think of** η **as measuring** ***reluctance to order*****?**

Implications of VW Axiom

- ***momentum*** **and** ***entropy balances*****:**

$$\dot{\mathbf{p}} = \mathbf{Div}\,\mathbf{S} - \mathbf{s} + \mathbf{d}, \quad \dot{\eta} = \mathbf{Div}\,\mathbf{h} - h + s \quad \textbf{in } \mathcal{P} \times I$$

- ***initial conditions*****:**

$$\mathbf{p}(x, t_i) = \mathbf{p}_i(x), \quad \eta(x, t_i) = \eta_i(x) \quad \textbf{for } x \in \mathcal{P}$$

- ***boundary conditions*** **on** $\partial\mathcal{P} \times I$**:**

$\mathbf{S}\mathbf{n} = \mathbf{c} \quad \equiv \quad$ **balance of contact forces:** $\mathbf{c} + \mathbf{S}(-\mathbf{n}) = \mathbf{0}$;

$\mathbf{h} \cdot \mathbf{n} = c \quad \equiv \quad$ **continuity cnd. on contact heating:** $c = (-\mathbf{h}) \cdot (-\mathbf{n})$

establishing $-\mathbf{h}$ **as a measure of specific** ***heat influx*** **at a point of an oriented surface of normal** $\mathbf{n}$**.**

Conservation of Internal Action. Preliminaries

An integral consequence of momentum and entropy balances is:

$$W(\mathcal{P}) + H(\mathcal{P}) = \frac{d}{dt}\Big(\int_{\mathcal{P}} (\mathbf{p}\cdot\mathbf{v} + \eta\,\vartheta)\Big) + \int_{\mathcal{P}} \textbf{\textit{stuff}},$$

where

- ***noninertial working***

$$W(\mathcal{P}) := \int_{\mathcal{P}} \mathbf{d}\cdot\mathbf{v} + \int_{\partial\mathcal{P}} \mathbf{c}\cdot\mathbf{v},$$

- ***heating***

$$H(\mathcal{P}) := \int_{\mathcal{P}} s\,\vartheta + \int_{\partial\mathcal{P}} c\,\vartheta\,,$$

- ***internal action***

$$\Phi(\mathcal{P}) := \int_{\mathcal{P}} \varphi, \qquad \textbf{with}$$

$$\textbf{\textit{stuff}} = \mathbf{s}\cdot\mathbf{v} + \mathbf{S}\cdot\nabla\mathbf{v} - \mathbf{p}\cdot\dot{\mathbf{v}} + h\,\vartheta + \mathbf{h}\cdot\nabla\vartheta - \eta\,\dot{\vartheta} := \dot{\varphi}.$$

The Axiom of Conservation of Internal Action

(CIA) ***In a cycle, the noninertial working plus the heating supplied to or extracted from $\mathcal{P}$ sum to null*****:**

$$\oint \big(W(\mathcal{P}) + H(\mathcal{P})\big) = 0\,.$$

Equivalently,

(CIA)′ ***In a cycle, the internal action is conserved*****:**

$$\oint \Phi(\mathcal{P}) = 0\,.$$

Implications of CIA Axiom. The 1st Law

For $\tau \equiv$ **specific *total energy*:**

$$\tau := \varphi + \mathbf{p}\cdot\mathbf{v} + \eta\,\vartheta, \quad T(\mathcal{P}) := \int_{\mathcal{P}} \tau,$$

we have:

$$\dot{T}(\mathcal{P}) = W(\mathcal{P}) + H(\mathcal{P}),$$ **the First Law of TD.**

To see this,

- **set** ***entropy inflow*** $(-\mathbf{h}, s)$ **proportional to** ***energy inflow*** $(-\mathbf{q}, r)$ **through** ***coldness*****:**

$$\mathbf{h} = \vartheta^{-1}\mathbf{q}, \;\; s = \vartheta^{-1}\, r\,;$$

- **accept standard notion of specific** ***kinetic energy*** κ**:**

$$\kappa := \frac{1}{2}\,\mathrm{p}\cdot\mathbf{v}, \quad \textbf{with} \;\; \dot{\mathrm{p}}\cdot\mathbf{v} = \mathrm{p}\cdot\dot{\mathbf{v}}, \;\; (\textbf{so that} \;\; \dot{\kappa} + (-\dot{\mathrm{p}})\cdot\mathbf{v} = 0)\,.$$

- **set**

$$\varepsilon := \tau - \kappa, \quad \varphi := \psi - \kappa,$$

and interpret

  - $\varepsilon \;\equiv$ **specific** ***internal energy***
  - $\psi \;\equiv$ **specific** ***Helmholtz free energy*** $= \varepsilon - \eta\,\vartheta$**,**

whence the interpretations for both the ***total energy*** τ **and the** ***internal action*** φ**.**

The Dissipation Axiom

(D) ***Whatever the process*** $(x,t) \mapsto \big(\mathbf{u}(x,t), \alpha(x,t)\big)$,

$$h\,\dot{\alpha} \leq 0$$

over the ***space-time cylinder*** $\mathcal{B} \times (0,T)$ **.**

Implications of D Axiom. The 2nd Law

- **Main implication is the generalized *dissipation inequality*:**

$$\dot{\psi} \leq -\eta\,\dot{\vartheta} + \mathbf{h} \cdot \nabla\vartheta + \mathbf{s} \cdot \mathbf{v} + \mathbf{S} \cdot \nabla\mathbf{v}\,.$$

- **If** $\vartheta \geq 0$ **(an unnecessary assumption so far),**
  then ***entropy balance & D axiom imply*:**

$$\dot{\eta} \geq \textbf{div}\,\mathrm{h} + s \quad (\textit{entropy inbalance} \equiv \textbf{ the Second Law of TD})\,.$$

- ***Standard dissipation inequality & entropy imbalance follow for***

$$\mathbf{h} = -\vartheta^{-1}\mathbf{q}\,, \quad s = \vartheta^{-1}\,r\,; \quad \mathbf{s} = \mathbf{0}.$$